Math 3450 - Test 2

Name: Solutions

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1. [16 points - 4 each] Fill in the rest of the definition.

(a) Let A and B be sets and $f : A \to B$. We say that f is one-to-one if

for every $a_1, a_2 \in A$ the following is true:
If $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$

could also have: If $f(a_1) = f(a_2)$, then $a_1 = a_2$.

(b) Let A and B be sets and $f : A \to B$. We say that f is onto if

range$(f) = B$ or for every $b \in B$ there exists $a \in A$ with $f(a) = b$

(c) Let A and B be sets and $f : A \to B$. Let $X \subseteq A$. We define the image of X under f to be

$f(X) = \{ f(x) \mid x \in X \}$

(d) Let A and B be sets and $f : A \to B$. Let $Y \subseteq B$. We define the inverse image of Y under f to be

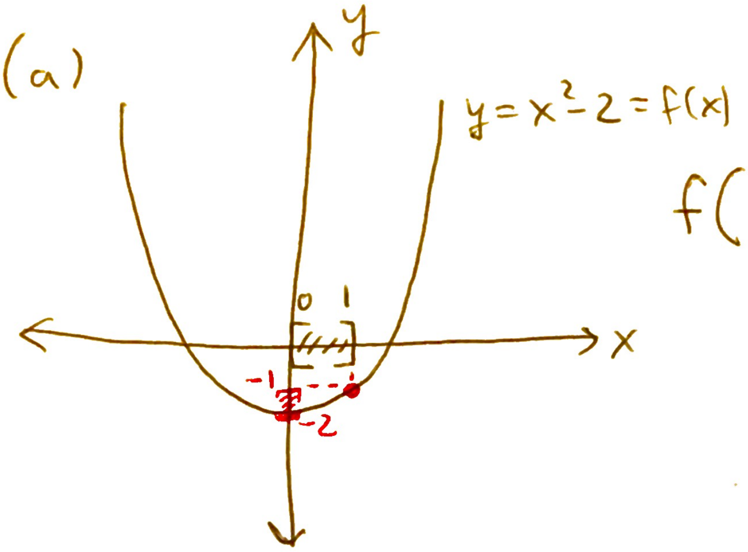
$f^{-1}(Y) = \{ a \in A \mid f(a) \in Y \}$

2. [10 points - 5 each] Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 - 2$.

(a) Compute $f([0, 1])$.

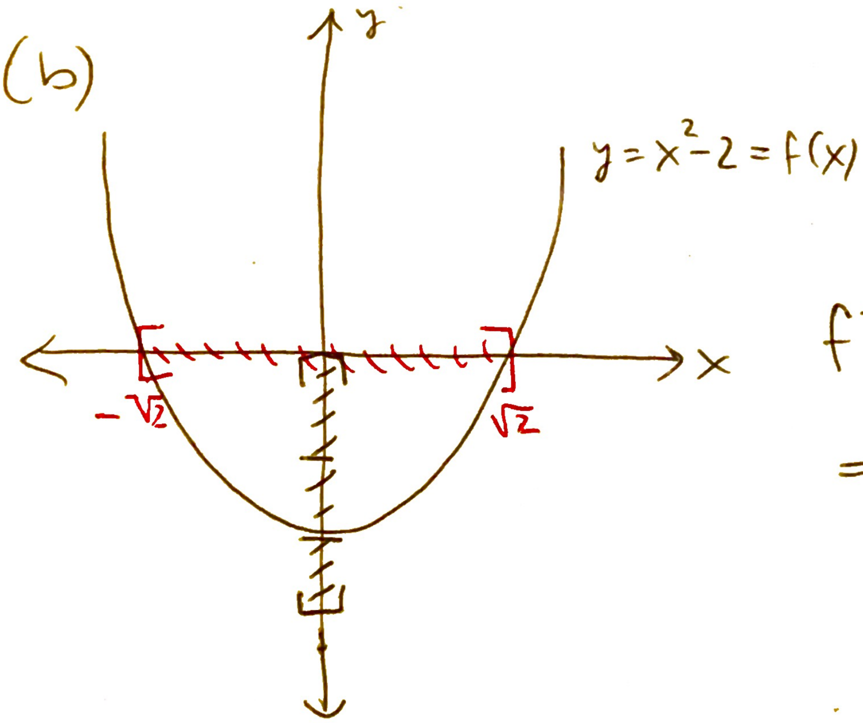
(b) Compute $f^{-1}([-3, 0])$.

(a)



$f([0,1]) = [-2,-1]$

(b)



$f^{-1}([-3,0])$

$= [-\sqrt{2}, \sqrt{2}]$

3. [20 points - 5 each] Let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ and $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be given by the formulas $f(m, n) = (m + n, n^3)$ and $g(m, n) = (2m + 1, n)$.

(a) Compute $g(0, 1)$ and also compute $(g \circ f)(1, 1)$.

$g(0,1) = (1,1)$

$(g \circ f)(1,1) = g(f(1,1))$
$= g(2,1) = (5,1)$

(b) Give a formula for $(g \circ f)(m, n)$.

$(g \circ f)(m,n) = g(f(m,n)) = g(m+n, n^3)$
$= (2m+2n+1, n^3)$

(c) Prove that g is one-to-one.

Suppose $g(m_1, n_1) = g(m_2, n_2)$ where $(m_1, n_1), (m_2, n_2)$ are in $\mathbb{Z} \times \mathbb{Z}$. Then $(2m_1+1, n_1) = (2m_2+1, n_2)$.
So, $2m_1 + 1 = 2m_2 + 1$ and $n_1 = n_2$.
Thus, $m_1 = m_2$ and $n_1 = n_2$.
So, $(m_1, n_1) = (m_2, n_2)$.

(d) Show that g is not onto.

Note that if $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ then $g(m,n) = (2m+1, n)$. And $2m+1$ is always odd. So you can never get an even integer in the 1st component of $g(m,n)$.

Ex: $(0,0) \notin$ range(g).
Suppose $g(m,n) = (0,0)$, where $(m,n) \in \mathbb{Z} \times \mathbb{Z}$.
Then $(2m+1, n) = (0,0)$.
So, $2m+1 = 0$.
Thus, $m = -\frac{1}{2} \notin \mathbb{Z}$.
Thus, there is no $(m,n) \in \mathbb{Z} \times \mathbb{Z}$ with $g(m,n) = (0,0)$. So, $(0,0)$ is not in the range of g and g is not onto $\mathbb{Z} \times \mathbb{Z}$.

4. [10 points] Pick <u>ONE</u> of the following. If you do both then I will grade A.

A) Consider the function $\pi_4 : \mathbb{Z} \to \mathbb{Z}_4$ given by the formula $\pi_4(x) = \overline{x}$. Let $Y = \{\overline{2}\}$. Prove that $\pi_4^{-1}(Y) = \{4k+2 \mid k \in \mathbb{Z}\}$.

B) Let $S = \mathbb{N} \times \mathbb{N}$. Define the relation $\sim$ on S where $(a,b) \sim (c,d)$ if and only if $a+d = b+c$. You can assume that $\sim$ is an equivalence relation, no need to prove it. Define the operation $\overline{(a,b)} \oplus \overline{(c,d)} = \overline{(a+c, b+d)}$. Prove that $\oplus$ is well-defined on the set of equivalence classes.

(A) This is similar to HW 4, 11(d).

$\underline{\pi_4^{-1}(Y) \subseteq \{4k+2 \mid k \in \mathbb{Z}\}}$

Let $x \in \pi_4^{-1}(Y)$.
Then $\pi_4(x) \in Y = \{\overline{2}\}$.
So, $\overline{x} = \overline{2}$.
Thus, $x \equiv 2 \pmod{4}$.
So, $x - 2 = 4k$ where $k \in \mathbb{Z}$.
So, $x = 2 + 4k \in \{4k+2 \mid k \in \mathbb{Z}\}$.

$\underline{\{4k+2 \mid k \in \mathbb{Z}\} \subseteq \pi_4^{-1}(Y)}$

Let $x \in \{4k+2 \mid k \in \mathbb{Z}\}$.
Then $x = 4k+2$ where $k \in \mathbb{Z}$.
So, $\pi_4(x) = \overline{x} = \overline{4k+2} = \overline{4}\,\overline{k} + \overline{2} = \overline{0} \cdot \overline{k} + \overline{2} = \overline{2} \in Y$.

$\boxtimes$

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B) Let $S = \mathbb{N} \times \mathbb{N}$. Define the relation $\sim$ on S where $(a, b) \sim (c, d)$ if and only if $a + d = b + c$. You can assume that $\sim$ is an equivalence relation, no need to prove it. Define the operation $\overline{(a, b)} \oplus \overline{(c, d)} = \overline{(a + c, b + d)}$. Prove that $\oplus$ is well-defined on the set of equivalence classes.

B) HW 3 #8(e).

5. [10 points] Pick ONE of the following. If you do both then I will grade A.

A) Let A and B be sets and $f : A \to B$. Prove that if $W \subseteq A$ and $Z \subseteq A$ then $f(W \cup Z) = f(W) \cup f(Z)$.

B) Let A, B, and C be sets and $f : A \to B$ and $g : B \to C$. (i) Prove that if f and g are both onto, then $g \circ f$ is onto. (ii) Prove that if f and g are both one-to-one, then $g \circ f$ is one-to-one.

A) HW 4 #14(a)

B) We proved these in class.
See the notes from Weds 10/23
